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The Stable Marriage Problem: An Extended Model for Three Parties

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Introduction

The original Stable Marriage Problem (SMP) is a famous problem in mathematics in which there are *n* men and *n* women, all of whom are to be paired with each other in a stable marriage.

A set of marriages is stable when no two people would rather be married to each other over their current spouses.

Instability:

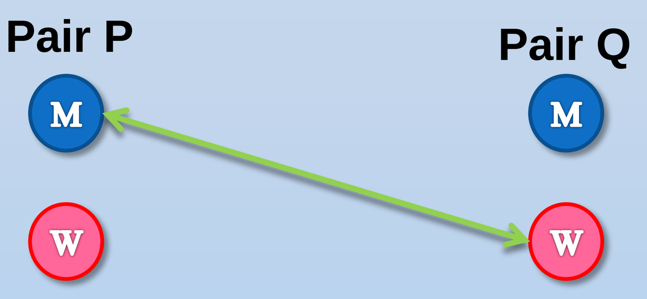

Figure 1 *This figure shows an unstable matching in which the man in pair P and the woman in pair Q prefer each other over their current partners*

Gale-Shapley (GS) algorithm

Men	Women
M1: W2 > W1 > W3	W1: M3 > M1 > M2
M2: W1 > W2 > W3	W2: M2 > M3 >M1
M3: W2 > W3 > W1	W3: M3 > M1 > M2

Table 2 *This table shows the preference list for each man and woman*

M1	M2	M3
~~W2~~	W1	W2
W1	~~W1~~	W2
W1	W2	~~W2~~
W1	**W2**	**W3**

Table 3 *This table shows each round of proposals in the GS algorithm when the men propose*

The SMP possesses several interesting properties such as:

- There is always at least one set of stable matches for all possible preferences.
- The set of matches constructed by the GS algorithm is the optimal stable set for all proposing people.
- The set of matches constructed by the GS algorithm is the pessimal stable set for all responding people.
- If the set of matches constructed by the GS algorithm when the men propose is the same as those of the women, the only set of stable matches is that set.

Extended Model

In our extended model, there are *n* people from each of the three different parties, all of whom are to be matched into stable threesomes.

A set of threesomes is stable when no person would rather be matched into a different threesome which he could be accepted in by both other members of the threesome is trying to switch into.

Instability:

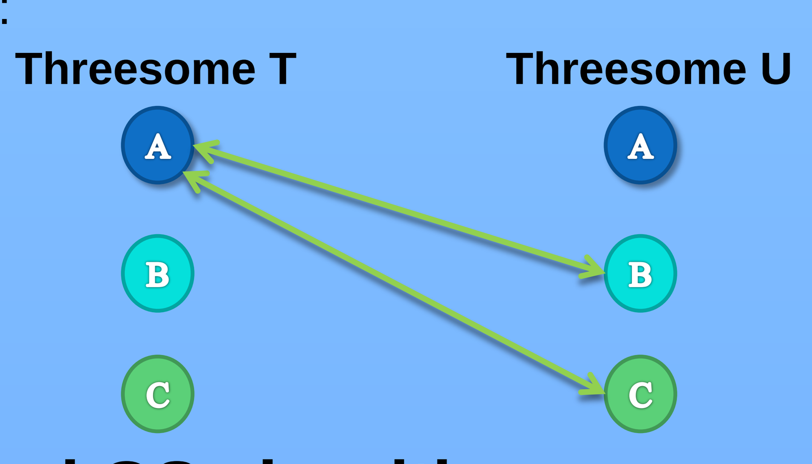

Figure 4 *This figure shows an unstable matching in which the A from threesome T and the B and C from threesome U prefer each other over their current partners from corresponding parties*

Modified GS algorithm

A1: B2 > B1 > B3 C1 > C3 > C2	B1: A3 > A1 > A2 C2 > C3 > C1	C1: A1 > A2 > A3 B2 > B3 > B1
A2: B1 > B2 > B3 C1 > C2 > C3	**B2:** A2 > A3 > A1 C1 > C2 > C3	**C2:** A2 > A3 > A1 B1 > B3 > B2
A3: B2 > B3 > B1 C3 > C2 > C1	**B3:** A3 > A1 > A2 C1 > C3 > C2	**C3:** A3 > A2 > A1 C1 > C2 > C3

Table 5 *This table shows the preference lists for each person.*

A1		A2		A3	
~~B2~~	C1	B1	~~C1~~	B2	C3
B1	C1	~~B1~~	C2	B2	C3
B1	**C1**	**B3**	**C2**	**B2**	**C3**

Table 6 *This table shows each round of proposals in the modified GS algorithm when the members of party A propose given the set of preference lists in Table 5.*

Properties

The modified GS algorithm for three parties maintains some, but not all of the properties of the original algorithm:

Stability: Because after every A, B and C is in a threesome, no A would want to allow a new member into the threesome who would at the same time want to be a member of that A's threesome, the resulting set of threesomes from running the modified GS algorithm is always stable.

Proposer Optimality: Unlike the results from the original GS algorithm, the optimal stable set of threesomes for each member of party P is not always the one generated when P is the proposing party. Consider the following counterexample:

A1: B1 > B2 C1 > C2	B1: A2 > A1 C1 > C2	C1: A1 > A2 B1 > B2
A2: B1 > B2 C2 > C1	**B2:** A2 > A1 C2 > C1	**C2:** A2 > A1 B2 > B1

Table 7 *This table shows the relevant preferences for each person*

Group A proposes: A1 – B2 – C1, A2 – B1 – C2 ; A1 is matched with favorite C and least favorite B.

Group C Proposes: A1 – B1 – C1, A2 – B2 – C2 ; A1 is matched with favorite C and favorite B.

Respondent Pessimality: By similar logic, the worst set of stable threesomes for each member of party R does not always occur when R is the responding party.

Number of Matches: When the stable set of matches received from the modified GS algorithm are the same regardless of whether A, B or C was the proposing party, there can still exist more than one stable set of threesomes. Consider the following counterexample:

A1: B1 > B2 > B3 C1 > C3 > C2	B1: A1 > A3 > A2 C1 > C2 > C3	C1: A1 > A2 > A3 B1 > B3 > B2
A2: B2 > B3 > B1 C2 > C1 > C3	**B2:** A2 > A1 > A3 C2 > C3 > C1	**C2:** A2 > A3 > A1 B2 > B1 > B3
A3: B3 > B1 > B2 C3 > C2 > C1	**B3:** A3 > A2 > A1 C3 > C1 > C2	**C3:** A3 > A1 > A2 B3 > B2 > B1

Table 8 *This table shows the preference list for each person*

In this example there are two stable sets. The first set contains threesomes in which all people get their first choices:

A1 – B1 – C1, A2 – B2 – C2, A3 – B3 – C3.

The second stable set contains threesomes in which all people get their second choice: ***A1 – B2 – C3, A2 – B3 – C1, A3 – B1 – C2***.

This set is also stable because no one can improve his or her situation by moving to another threesome since no person prefers both members of alternative threesomes than his own.

Take A2 as an example:

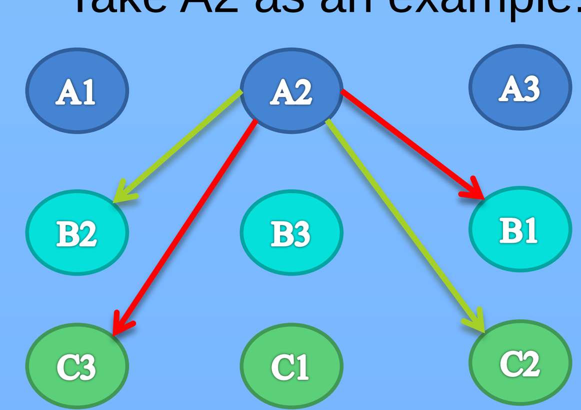

Figure 9 *This figure depicts A2's inability to switch into other threesomes. Green arrows signify A2's preference of a person over his current partner, while red arrows demonstrate A2's preference of his current partner.*

Discussion

We explored several properties of Stable Marriage Problem, such as guaranteed stability and the quality of matches for the proposing and responding parties, and examined the ramifications of adding a third party. We found that the extended model with three parties shares some interesting characteristics with the original model, while differing in other respects.

Acknowledgements

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